International Journal of Theoretical Physics, Vol. 4, No. 5 (1971), pp. 361–388

Group Deformations and Relativity

YU. A. ROGOZHIN

Institute of Physics and Energetics, Obninsk, USSR

Received: 17 *June* 1970

Abstract

A theory of deformation (homeomorphism but non-isomorphism) of topological groups is developed. In particular, a theory of deformation of subgroups structure is considered. The whole formalism is based on conceptions of holonomicity and relative geometry.

A field is postulated to deform the symmetry group of a free physical system. It is shown that the classical fields deform the Poincare group. Thanks to this fact, gravitation appears as space-time curvature (non-holonomicity of the Lorentz subgroup mapping); and electromagnetism reveals itself by space-time torsion (non-holonomicity of the translation subgroup mapping). From physically evident premises it follows that space-time also has a torsion in the rotating and accelerated systems of reference.

1. *Introduction*

In spite of significant progress and rapid development of group-theoretical methods in physics, field description in frames of this formalism is still an unsolved problem. Reference to the theory of compensating fields (Utiyama, 1956; Sakyrai, 1960) can only affirm this statement. The difficulty is that the theory of compensating fields deals with a Lagrangian which firstly is not a group-theoretical construction and secondly it has been criticised for various other reasons (Chew, 1961). The remarks made concern all the Lagrangian methods of description of interactions. The following question may arise in this connection: 'Is it possible, whilst staying in the frames of group-theoretical formalism, being based only on the equations of motion in a field, to say something essential about the field itself?'

Let the state of a physical system have a symmetry group under space-time, dynamical and isotopical variables when there is not an interaction, or when one neglects it. Then it is obvious that the interaction breaks down some of these symmetries, to 'eliminate corresponding degeneracies of the physical system'. This results in two means for the description of interactions.

In the first place one can study 'contraction' of a free-system symmetry group into a group of the (retained) symmetries of the interacting system.

Secondly, a free-system symmetry group 'deformation' can be investigated in a group including the rest of the symmetries.

Usually, one chooses the first way, in which the interaction itself is excluded from group-theoretical consideration. To compensate for this, Lagrangian, Hamiltonian or S-matrix, invariant under some symmetry group, are postulated. That is why an answer to the question is possible only on the basis of the second technique, to which less attention has been paid.

Most likely, this is explicable by the absence of an effective theory of arbitrary groups. In the case considered, the problem is somewhat simplified if one implies a physical field as an interaction defined on space-time (Streater & Wightman, 1964).† Here it is necessary to study a deformation of free space-time motion group into an arbitrary topological group.

Unfortunately, one has to refuse both the theory of differentiable manifolds (Bishop & Critenden, 1964) (because of too strict demands of differentiability) and the theory of fiber bundles (due to the 'parallel displacement' concept and other essentially non-group theoretical constructions on which this theory is based).

It then becomes clearer why the major part of the work suggested, concerns the development of the mathematical aspects of group deformation theory which allows one to describe properties of a given group relative to a topologically equivalent Lie group. Hence the concept of 'relative geometry' naturally arises.

At the next stage, the immersion of the group deformation theory in physics gives the possibility of formulation of the theory of relativity as a relative geometry of a given field.

To illustrate this, attempts are made to show that the classical theory of relativity can be considered as the relative geometry of the Poincare group deformation. Besides curvature, a torsion also appears here, which manifests itself in an accelerated reference system and is due to the electromagnetic field.

2. *Holonomicity of Topological Groups*

Speaking about topological groups we shall assume everywhere in this paper that they are linearly connected. It means that any element G from the topological group $\mathcal{G}$ can be connected with the unit E through a one-parameter subgroup $\mathcal{G}_\sigma$ which is given by some continuous mapping:

$$G = G(\chi^0, \chi^1, \ldots, \chi^{\kappa-1}) \tag{2.1}$$

The mapping $G(\cdot)$ is determined on a curve $\chi(\sigma)$ in the number space $\mathfrak{H} = R^\kappa$.

It follows from definition of the subgroup $\mathcal{G}_\sigma$ that for any $\chi = \chi(\sigma)$ and $\xi = \chi(\sigma')$ a point $\zeta = \chi(\sigma'')$ can be found on the same curve providing

$$G(\zeta) = G(\xi) \cdot G^{-1}(\chi) \tag{2.2}$$

† This interaction is defined immediately at each point or by means of functions defined at each point of space-time.

where

$$G(\zeta) \to E \quad \text{if} \quad \xi \to \chi \tag{2.3}$$

The mapping $G(\cdot)$ will be called formation of the subgroup $\mathscr{G}_a$ and K-dimension vector space $\mathfrak{H}$ will be parametric space. When $\mathscr{G}$ is a Lie group the formation of its any subgroup may be extended to differentiable mapping of the parametric space proper onto the whole group (Pontryagin, 1959). Since the continuous symmetries are described by Lie groups, a breakdown of symmetries must break down the formation differentiability. The following effects will be investigated below.

A small translation (differential) $d\chi$ in parametric space $\mathfrak{H}$ induces the differential of formation

$$DG(\chi) = G(\chi + d\chi) - G(\chi) \tag{2.4}$$

and the differential of the group which will be defined as follows

$$\bar{d}G(\chi) = G(\chi + d\chi).G^{-1}(\chi) \tag{2.5}$$

Some differentiation rules for the topological groups are deduced in the Appendix. It is useful to introduce the truncated differential dG as

$$\bar{d}G(\chi) = E + dG(\chi) \tag{2.6}$$

where for an additive group

$$E \equiv 0; \qquad dG = \bar{d}G = DG \tag{2.6a}$$

and for a multiplicative one

$$dG(\chi) = DG(\chi).G^{-1}(\chi) \tag{2.6b}$$

Due to the continuity of formation of the $\mathscr{G}$ and with $d\chi$ small, the truncated differential of group dG is approximated to by the linear function of parameter differential

$$dG(\chi) \approx G_i(\chi).d\chi^i \tag{2.7}$$

where G_i will be called a generator of the group. All the properties of the group are defined locally at a given value of parameters and can be different at different points of parametric space $\mathfrak{H}$.†

If $\mathscr{G}$ is a Lie group, then for additive and multiplicative group structure, in view of (2.6) and (2.7) we have

$$G_i^{ad} = \frac{\partial G}{\partial \chi^i}; \qquad G_i^{m} = \frac{\partial G}{\partial \chi^i} G^{-1} \tag{2.8}$$

from which analyticity of Lie group generators follows. $G_i(0) \equiv G_i^0$ is called an infinitesimal operator of group (Neimark, 1963).

† Later, this will always be meant, though corresponding marks will often be omitted.

Let us now consider a holonomy operator in the topological group $\mathscr{G}$:

$$[\delta;d]G(\chi) \equiv \delta G^{-1}(\chi).dG^{-1}(\chi+\delta\chi).\delta G(\chi+d\chi).dG(\chi) \qquad (2.9)$$

where $\delta\chi$, $d\chi \in \mathfrak{H}$; $dG^{-1} = (dG)^{-1}$. Since the transitivity of the group in eigenspace means $[\delta;d]G \in \mathscr{G}$ and because of (2.5)

$$G(\chi+d\chi) = dG(\chi).G(\chi) \qquad (2.10)$$

then it is easily seen that the holonomy operator $[\delta;d]$ determines the change of the group: $F(\chi) = [\delta;d]G(\chi).G(\chi)$ when a (closed) circuit in parametric space has been passed. A set of holonomy operators $\mathscr{V} \ni [\delta;d]$ forms, as E. Cartan (1927a) noted, a group† which, as will be shown below, is a mapping group of Lie algebra of Lie group.

To prove this, an approximation to (2.9) must be made with the accuracy up to the second-order of δG, dG. Using the formulas of the Appendix, namely, substituting (A4) ÷ (A6), (A8) ÷ (A11) into (2.9), we have a simple expression for a holonomy operator

$$[\delta;d]G = E + [\delta G, dG] - [\delta, d]G + O_3 \qquad (2.11)$$

where all the values are already taken at one point $\chi \in \mathfrak{H}$; O_3 includes the terms of the third and higher orders of δG, dG; the commutator brackets act by the usual rules:

$$[\delta G, dG] = \delta G.dG - dG.\delta G; \qquad [\delta, d]G = \delta dG - d\delta G \qquad (2.12)$$

Substitution of (2.7) and (A7) into (2.11) gives a generator of the group $\mathscr{V}.\mathscr{G}$.‡

$$G_{[i;k]} = [G_i, G_k] + G_{[i,k]} \qquad (2.13)$$

where $[G_i, G_k] = G_iG_k - G_kG_i$ is a commutator of group generators, being an element of the corresponding Lie algebra (Pontryagin, 1959); $G_{[i,k]} = G_{i,k} - G_{k,i}$ is the outer derivative which also satisfies the axioms of Lie algebra (Cartan, 1927a). Since $G_{[i;k]}$ belongs to a Lie algebra, then due to (2.13) the above statement has been proved.

Taking into account this fact, we say that the holonomy generator $G_{[i;k]}$ generates holonomy algebra $\mathscr{V} \circ \mathscr{G}$ of the topological group by differentials $[\delta;d]G \in \mathscr{V} \circ \mathscr{G}$:

$$[\delta;d]G = G_{[i;k]}\delta\chi^i d\chi^k \qquad (2.14)$$

(outer differentials of group) onto the field $\mathfrak{H} \times \mathfrak{H}$, where $d\chi \in \mathfrak{H}$. The correspondence between algebra $\mathscr{V} \circ \mathscr{G}$ and group $\mathscr{V}.\mathscr{G}$ is trivial. Let us call the topological group holonomic if the corresponding holonomy algebra consists only of the zero element (which means the absence of a

† It was named by Cartan the 'holonomy group'.

‡ Here one supposes holonomicity of parametric space: $[\delta, d]\chi^i = 0$.

change in the group after the circuit in parametric space has been passed), i.e.

$$[\delta;d]G = G_{[i;k]} \equiv 0; \qquad [\delta;d]G \equiv E \tag{2.15}$$

for any $\chi \in \mathfrak{H}$ and $[\delta;d] \in \mathscr{V}$.

The importance of the concepts introduced illustrates

Theorem 1: Lie group is holonomic.

For the additive group it is obvious from (2.8). For multiplicative Lie group

$$G_{i,k} = \frac{\partial^2 G}{\partial\chi^i \partial\chi^k} G^{-1} + \frac{\partial G}{\partial \chi^i} \cdot \frac{\partial G^{-1}}{\partial \chi^k} \tag{2.16}$$

But from $GG^{-1} = E$ it follows that

$$\frac{\partial G^{-1}}{\partial \chi^k} G + G^{-1} \frac{\partial G}{\partial \chi^k} = 0; \qquad \frac{\partial G^{-1}}{\partial \chi^k} = -G^{-1} \frac{\partial G}{\partial \chi^k} G^{-1} \tag{2.17}$$

Substitution of (2.17) into (2.16) with respect to (2.8) gives:

$$G_{[i,k]} = -[G_i, G_k] \tag{2.18}$$

This proves the theorem.

Thereby it becomes reasonable to name the element of holonomy algebra defined from (2.13) a twist of the group (twist of holonomic group is absent).

In Lie finite-dimensional algebra (Jacobson, 1961) there exists a local basis in which

$$[G_i, G_k] = c^m_{ik} G_m; \qquad G_{[i,k]} = \mathscr{C}^m_{ik} G_m \tag{2.19}$$

whereas†

$$c^m_{ik} = -c^m_{ki}; \qquad c^m_{ik} c^n_{ml} + c^m_{kl} c^n_{mi} + c^m_{li} c^n_{mk} = 0 \tag{2.20}$$

(analogous correlations must be satisfied for $\mathscr{C}^m_{ik}$). As soon as the local basis may change from point to point of parametric space, then all the quantities in (2.19) can depend on $\chi \in \mathfrak{H}$. Naturally one can ask a question: 'Under which conditions do the structural functions $c^m_{ik}(\chi)$, $\mathscr{C}^m_{ik}(\chi)$ transform into structural constants?' An answer to this question results in:

Theorem 2. A holonomic group is defined by structural constants and infinitesimal operators.

Let the correlations of (2.19) be fulfilled at point $\chi \in \mathfrak{H}$. Then at a neighbouring point $\chi + d\chi$ the following takes place:

$$[G_i(\chi + d\chi), G_k(\chi + d\chi)] = c^m_{ik}(\chi + d\chi) G_m(\chi + d\chi) \tag{2.21}$$

† Here and thereafter we shall define elements of the group (implying matrix representation) by capital letters; small letters will refer to numbers, functionals and matrix elements (with corresponding indices). Vector $\chi \equiv (\chi^0, \chi^1, \ldots, \chi^k)$ in parametric space will be an exception.

Because of (A3) and (A4)

$$E + \delta G(\chi + d\chi) = [E + d\delta G(\chi)].[E + \delta G(\chi)]$$

Whence

$$G_i(\chi + d\chi) = G_i + G_{i,k} d\chi^k + G_{i,k} G_m d\chi^k d\chi^m + \cdots \tag{2.22}$$

where all the quantities in the right-hand side are determined at point $\chi \in \mathfrak{H}$. Substituting (2.22) into (2.21) we obtain

$$[G_i(\chi + d\chi), G_k(\chi + d\chi)] = [G_i, G_k] + [G_{i,m}, G_k] d\chi^m + [G_i, G_{k,m}] d\chi^m + O_2 \tag{2.23}$$

where O_2 comprises outer differential forms (Cartan, 1927) of higher order. These forms can be expressed as superpositions of various degrees of the second and third members in the right-hand side of (2.23). Thus Theorem 2 demands the fulfilment of equalities

$$[G_{i,m}, G_k] + [G_i, G_{k,m}] = 0 \tag{2.24}$$

for any values of indices. Having made double-cycle permutation of indices in (2.24) and subtracting equations obtained from the initial one, we have

$$[G_{[i,m]}, G_k] + [G_{[m,k]}, G_i] + [G_{[k,i]}, G_m] = 0 \tag{2.25}$$

This, in view of (2.19), is equivalent to the equation

$$\mathscr{C}^n_{im} c^l_{nk} + \mathscr{C}^n_{mk} c^l_{ni} + \mathscr{C}^n_{ki} c^l_{nm} = 0 \tag{2.26}$$

For the holonomic group $\mathscr{C}^l_{km} = -c^l_{km}$ and equations (2.26) and (2.20) is automatically satisfied in all points $\chi \in \mathfrak{H}$ whereby the statement of theorem has been proved.

It should be noted that the equation (2.26) has been satisfied when $\mathscr{C}^l_{km}(\chi) = \epsilon c^l_{km}(\chi)$, where ϵ is any number. From the above considerations it follows: $[G_i(\chi), G_k(\chi)] = \text{const}$, whence by the formula (2.17) we get $G_{[i;k]}(\chi) = (\epsilon + 1)[G_i(\chi), G_k(\chi)] = \text{const}$. The uniformly twisted groups (which Cartan named 'symmetric spaces' (Cartan, 1927) meaning first of all the spaces of constant curvature), as well as the holonomic groups, are defined by the structural constants and the infinitesimal operators. But the uniformly twisted groups (and groups with constant generators in their number) are not the Lie groups, being nonholonomic groups.

Theorem 2 allows one to extend all thoroughly developed formalism of Lie groups to any holonomic groups.

But for our purposes this formalism is not all we need, because the task put in the Introduction makes it necessary (as it will be shown later on) to study the deformation of Lie groups into non-holonomic topological groups.

3. *Deformation of Topological Groups*

Let us consider homeomorphism $\varphi : \mathfrak{H} \to \mathfrak{H}'$ of parametric space

$$\varphi(\chi) = \bar{\chi}; \qquad \varphi^{-1}(\bar{\chi}) = \chi \tag{3.1}$$

to which corresponds linear reversible mapping of differentials

$$d\chi^l = \varphi_k{}^l d\bar{\chi}^k; \qquad d\bar{\chi}^k = \bar{\varphi}_m{}^k d\chi^m; \qquad \varphi_k{}^l \bar{\varphi}_m{}^k = \delta_m{}^l \tag{3.2}$$

but $d\bar{\chi}$ does not need to be a full differential, and $\mathfrak{H}'$ may be non-holonomic.

Homeomorphism φ induces homeomorphism $\Psi: \mathscr{G} \to \mathscr{G}'$ of group $\mathscr{G}$ into group $\mathscr{G}'$ (homeomorphism of group formation):

$$G'(\bar{\chi}) = \Psi\{G[\varphi(\chi)]\} = G(\chi) \tag{3.3}$$

Since homeomorphism groups are reciprocally continued and one-to-one valued one can put reciprocal correlation between the neighborhoods of units $V(E) \in \mathscr{G}$ and $V'(E') \in \mathscr{G}'$. Contracting these neighborhoods to the point one obtains

$$E' = E \tag{3.4}$$

Because $d\chi$ is small and homeomorphism is continued then

$$\varphi(\chi + d\chi) = \bar{\chi} + d\bar{\chi} + O_2 \tag{3.1a}$$

where O_2 is the remainder of second-order of $d\chi$. As the process is continued

$$\Psi\{G[\varphi(\chi + d\chi)]\} = \Psi[G(\bar{\chi} + d\bar{\chi} + O_2)] = G'(\bar{\chi} + d\bar{\chi}) + O_2 \tag{3.5}$$

On the other hand, due to (3.3)

$$\Psi\{G[\varphi(\chi + d\chi)]\} = G(\chi + d\chi) \tag{3.6}$$

Comparing the last two expressions, with respect to (2.10), it is possible to deduce, first of all, that homeomorphism is locally homomorphic

$$dG'(\bar{\chi}).G'(\bar{\chi}) = \Psi[dG(\bar{\chi}).G(\bar{\chi})] = dG(\chi).G(\chi) \tag{3.7}$$

Secondly, as $G(\chi)$ in (3.7) is arbitrary and considering (3.3), it follows that

$$dG'(\bar{\chi}) = d\Psi[G(\bar{\chi})] = dG(\chi) \tag{3.8}$$

from whence in view of (2.6), (2.7), (3.2) and (3.4) one gets linear interrelation of the generators of homeomorphic groups:

$$G_l'(\bar{\chi}) = \varphi_l{}^k G_k(\chi) \tag{3.9}$$

The latter result readily gives linear transformation of commutators

$$[G_l', G_k'](\bar{\chi}) = \varphi_l{}^m \varphi_k{}^n [G_m, G_n](\chi) \tag{3.10}$$

However, as

$$G'_{l;k}(\bar{\chi}) = \varphi^m_{l,k}(\chi)\, G_m(\chi) + \varphi_l{}^m(\chi)\varphi_k{}^n(\chi)\, G_{m,n}(\chi) \tag{3.11}$$

then generally non-linear mapping of outer derivatives follows

$$G'_{[l,k]}(\bar{\chi}) = \varphi^m_{[l,k]}\, G_m(\chi) + \varphi_l{}^m \varphi_k{}^n\, G_{[m,n]}(\chi) \tag{3.12}$$

Corresponding change of group twist

$$G'_{[l;k]}(\bar{\chi}) = \varphi_l{}^m \varphi_k{}^n\, G_{[m;n]}(\chi) + \varphi^m_{[l,k]}\, G_m(\chi) \tag{3.13}$$

can be interpreted as a non-linear transformation of structure functions

$$c'^{l}_{km} = \varphi_a{}^l \varphi_k{}^b \varphi_m{}^c c^a_{bc}; \qquad \mathscr{C}'^{l}_{km} = \varphi_a{}^l \varphi_k{}^b \varphi_m{}^c \mathscr{C}^a_{bc} + \varphi^l_{[k,m]} \tag{3.14}$$

If Θ is an automorphism (a transformation of coordinates) of parametric space then $\theta_k{}^l = (\partial\chi^l/\partial\bar{\chi}^k)$ and $\theta^l_{[k,m]} = 0$ so that isomorphism of holonomy algebra is induced as on the basis of (2.14), (3.2) and (3.13)

$$[\delta; d]\, G'(\bar{\chi}) = [\delta; d]\, G(\chi) \tag{3.15}$$

From this and from the definition of holonomy operator it follows that Θ is (globally) homomorphic and, consequently (Pontryagin, 1959), the homeomorphic groups $\mathscr{G}$ and $\Theta(\mathscr{G})$ are isomorphic.

Vice versa, if groups $\mathscr{G}$ and $\mathscr{G}'$ are isomorphic they are homomorphic and homeomorphic. That is why holonomy algebras $\mathscr{V} \circ \mathscr{G}$ and $\mathscr{V} \circ \mathscr{G}'$ must be isomorphic.

Thus

$$\varphi^l_{[k,m]} = 0 \tag{3.16}$$

are necessary and sufficient conditions for group isomorphism determined by (3.2), (3.3).

Let us call group homeomorphism $\Psi: \mathscr{G} \to \mathscr{G}'$, determined with the help of differentiable functions $\varphi_k{}^l(\chi)$ in correlations (3.9) among group generators, a deformation of group $\mathscr{G}$ into group $\mathscr{G}'$ if $\varphi^l_{[k,m]} \neq 0$.† A set of deformations forms a group of topological transformations; in its turn, a set of the topological groups, deformed one into another, forms a topological class in which deformations act.

It has been shown that:

Theorem 3: Within the topological class a group is determined by its holonomy algebra with accuracy up to isomorphism.

It follows directly from Theorem 3 that the topological class comprises not more than one holonomic group (in particular owing to Theorem 1, not more than one Lie group).

The group deformation was defined above by means of non-holonomic homeomorphism φ of parametric space. Nevertheless, sometimes the description of group deformation, from the viewpoint of the theory of representations, with the help of induced homeomorphism of the groups themselves appears more convenient. For this purpose (3.3) will be written in the following form:

$$\Psi[G(\chi)] = G'(\chi) = S'(\chi) . G(\chi) \tag{3.17}$$

where $S(\chi)$ can be called the deformation operator.

† 'Contraction' and 'deformation' of Lie algebras, intensively investigated in recent years (Ihonu & Wigher, 1953; Gerstenhaber, 1964; Levy-Nahas, 1967), are limited by reciprocal mapping of Lie groups.

As far as group mapping: $\mathscr{G} \to \mathscr{G}'$ is homeomorphic the operator S is reversible. Also, the sequence of two homeomorphisms (non-holonomic ones in their number) $\varphi_2[\varphi_1(\chi)]$ of parametric space can always be substituted by one homeomorphism $\varphi_{21}(\chi)$. Therefore operators S_1 and S_2 are associative and hence form group stricture (a group deformations). Differentiation of (3.17) in accordance with (A.14) and (A.17) gives:

$$dG' = dS + S.dG.S^{-1} \tag{3.18}$$

$$\delta dG' = \delta dS + S\{\delta dG + [\delta S, dG]\} S^{-1} \tag{3.19}$$

Substitution (3.18) and (3.19) into (2.15) results in

$$[\delta;d]G' = [\delta;d]S + S.[\delta;d]G.S^{-1} \tag{3.20}$$

where all the values are determined at the point $\chi \in \mathfrak{H}$.

Thus for homeomorphism Ψ to be isomorphism (transformation of coordinates in the group) it is necessary and sufficient that the condition

$$[\delta;d]S(\chi) \equiv 0 \tag{3.21}$$

should be fulfilled.

We shall say that formula (3.3) is a horizontal description of homeomorphism Ψ, and (3.17) is its vertical description. In general, an arbitrary homeomorphism $\Psi:\mathscr{G} \to \mathscr{G}'$ can be described as a sequence $\Psi = \Omega \circ \Sigma$ of the horizontal homeomorphism Σ: $G''(\bar{\chi}) = G(\chi)$ and the vertical homeomorphism $\Omega: G'(\bar{\chi}) = S(\bar{\chi}).G''(\bar{\chi})$. It results in

$$G'(\bar{\chi}) = S(\bar{\chi}).G(\chi) \tag{3.22}$$

so that the form of the equations (3.17) + (3.20) is held, but the quantities included in them would be determined, in the light of (3.22) above various parametric spaces. Therefore, because of (3.2) and (3.18) a nonlinear mapping of generators is possible:

$$G_i'(\bar{\chi}) = S_i(\bar{\chi}) + \varphi_i^{\,k} S.G_k(\chi).S^{-1} \tag{3.23}$$

4. *Deformation of Subgroup Structure*

In the above section it is shown that the deformation is revealed in a disturbance of the group structure. The disturbance of subgroup structure of the topological group due to deformation is of special interest, but the construction of corresponding general theory is not our task. That is why we shall study here only the deformation of the group structure dealing with two subgroups.†

† The general theory of subgroup structure deformation is likely to be constructed with induction from this particular case without substantial difficulty.

Let $\mathscr{L} \ni L$ and $\mathscr{Y} \ni Y$ be two topological groups, whereas

$$dY(\alpha) = E_Y + Y_i(\alpha)\,d\alpha^i; \qquad i = 0, 1, \ldots, m$$
$$dL(\beta) = E_L + L_{(i)}(\beta)\,d\beta^{(i)}; \qquad (i) = (0), (1), \ldots, (h) \tag{4.1}$$

so that each group owes its own parametric space:

Let us consider now the topological group $\mathscr{P} \ni P$ formed by a union of two groups: $P = L \vee Y$ so that

$$E_P = E_L \vee E_Y; \qquad P_\alpha = L_\alpha \vee Y_\alpha; \qquad P_{\alpha,\beta} = L_{\alpha,\beta} \vee Y_{\alpha,\beta} \tag{4.2}$$

Parametric space $\mathfrak{H}_P$ is a product of constituent group parametric spaces, thereby the indices in (4.2) go through the whole set of i, (i) values.† Let us demand that the union operation be locally linear and distributive:

$$(adL' + bdL'') \vee (adY' + bdY'') = a(dL' \vee dY') + b(dL'' \vee dY'') \tag{4.3}$$

The rest of its features are preset or have been determined from group structure:

$$dP = dP'.dP'' = (dL' \vee dY').(dL'' \vee dY'') = dL \vee dY \tag{4.4}$$

The constituent groups $\mathscr{Y}$ and $\mathscr{L}$ would intersect in the compound group $\mathscr{P}$ providing an appearance of supplementary generators:

$$Y_{(i)} = \chi^{k}_{.(i)}\, Y_k \tag{4.5}$$

$$L_i = \chi^{(k)}_{.i}\, L_{(k)} \tag{4.6}$$

Together with basic generators the latter forms a full system of $\mathscr{P}$ group generators. Union (4.2) of these generators, with respect to (4.3) and (4.5) defines the holonomy algebra $\mathscr{V} \circ \mathscr{P}$. Thus, because of Theorem 3, subgroup structure given by the above operations of union and intersection‡ of the subgroups defines the group with accuracy up to isomorphism within the given topological class.

Linear mapping (3.9) of generators by horizontal homeomorphism provides on the basis of (4.2) and (4.3) invariance of subgroup union

$$\Psi(dL \vee dY) = \Psi(dL) \vee \Psi(dY) \tag{4.7}$$

due to which

$$L_\alpha'(\bar\chi) = \varphi_\alpha{}^{\gamma}(\chi) L_\gamma(\chi); \qquad Y_\alpha'(\bar\chi) = \varphi_\alpha{}^{\gamma}(\chi)\, Y_\gamma(\chi) \tag{4.8}$$

so that in view of (4.5)

$$L_i'(\bar\chi) = (\varphi_i{}^{m} \chi^{(a)}_{.m} + \varphi_i^{(a)}) L_{(a)}(\chi) \tag{4.9}$$

† In other words, the group $\mathscr{P}$ contains two subgroups $E \vee \mathscr{Y}$ and $\mathscr{L} \vee 0$, $\mathfrak{H}_Y \times \mathfrak{H}_L = \mathfrak{H}_P$ being its parametric space. Thus $Pr_i \chi = \chi^i = \alpha^i \in \mathfrak{H}_Y$ and $Pr_{(i)} \chi = \chi^{(i)} = \beta^{(i)} \in \mathfrak{H}_L$.

‡ Subgroup structure is usually defined in some other way (Suzuki, 1956) through the operations of Boolean algebra of sets. This way makes it possible to construct a more abstract but more complicated theory.

On the other hand,

$$L_i' = \bar{\chi}_{.i}^{(k)} L'_{(k)} = \bar{\chi}_{.i}^{(k)} (\varphi_{(k)}^{m} \chi_{.m}^{(a)} + \varphi_{(k)}^{(a)}) L_{(a)} \tag{4.10}$$

Equating expressions (4.9) and (4.10) one obtains, because of $L_{(a)}$ independence

$$\bar{\chi}_{.i}^{(k)} (\varphi_{(k)}^{m} \chi_{.m}^{(a)} + \varphi_{(k)}^{(a)}) = \varphi_i^{m} \chi_{.m}^{(a)} + \varphi_i^{(a)} \tag{4.11}$$

Thereby it is shown that group homeomorphism changes subgroups intersection (non-linearly). If the subgroups are not intersected in the initial group, i.e. $\chi_{.m}^{(a)} \equiv 0$, then

$$\bar{\chi}_{.i}^{(k)} \varphi_{(k)}^{(a)} = \varphi_i^{(a)} \tag{4.12}$$

Similarly one can consider homeomorphism of the other subgroup $\Psi(Y)$. In particular having $\chi_{.(k)}^{i} \equiv 0$, one obtains the equations, 'conjugated' with (4.12)

$$\bar{\chi}_{.(i)}^{k} \varphi_k^{a} = \varphi_i^{(a)} \tag{4.13}$$

in which the indices interchanged the brackets.

Thus invariance of subgroup union under group homeomorphism by no means provides invariance of the subgroups themselves because of subgroup intersection non-invariance in general: $dL'(\chi) \neq dL(\chi)$.

As far as

$$\varphi_{[i,k]}^{(a)} = \bar{\chi}_{.i}^{(m)} \varphi_{(m),k}^{(a)} - \bar{\chi}_{.k}^{(m)} \varphi_{(m),i}^{(a)} + \bar{\chi}_{[i,k]}^{(m)} \varphi_{(m)}^{(a)} \tag{4.14}$$

then the reduction to zero of left-hand side parts of equations (4.14) and those of 'conjugated' with them is, due to (3.16), a condition necessary for group isomorphism: $\mathscr{P} \to \mathscr{P}'$. Therefore it is immediately seen that group isomorphism does not require isomorphism of subgroup structure. On the other hand the isomorphism of subgroup structure is insufficient for group isomorphism and conditions of absence of the subgroup deformation as was already mentioned, must be added:†

$$\varphi_{[k,m]}^{i} = \varphi_{[(k),(m)]}^{(i)} \equiv 0 \tag{4.15}$$

It should be noted that the above horizontal description of the subgroups structure does not depend on the production law (4.4) which is to be adaccounted for in the vertical description. Nevertheless, the definitions of union (4.2) and of intersection (4.6) of the subgroups, as well as the general way of discussion are retained.

5. *Relative Geometry and the Theory of Relativity*

The chief problem in geometry was formulated by Klein in his Erlangen program (Klein, 1893) as follows:

'A manifold and a group of transformations in it are given. The invariant theory of the group is to be developed'.

† In the canonical theory (Suzuki, 1956) group isomorphism governs the isomorphism of subgroup structure and not vice versa.

This means that the geometrical methods can be applied to the investigation of only such topological groups the eigenspace of which (the space of the given group formation images) forms a manifold. Since, through the definition (Chevalley, 1946), the manifold is prescribed by some family of analytical functions on k-dimension affine space then it is obvious that the 'geometrical' groups must have analytical formation, i.e. they must be Lie groups.

It may somehow be possible to expand the class of geometrical groups if the requirements on the functions prescribing the manifolds are reduced to (a sufficient number of times) differentiability. The corresponding group is still a holonomic one.

The last condition—holonomicity—is essential for the group 'geometricity'. In fact, the coordinates of non-holonomic group elements are non-invariant under the simplest motions (closed cycles) of its parameters. Therefore the non-holonomic group, generally speaking, does not have any invariances.

Cartan (1927b) often stressed the meaning of the holonomicity concept for the investigation of group geometry. Nevertheless, the synthesis of Erlangen program ideas with Riemannian geometry ideas was carried out by him (Cartan, 1927a) and his successors (Lichnerowicz, 1955; Nomizu, 1956) in the theory of fiber bundles not on group-theoretical but on a geometrical basis with the help of the concept of parallel displacement.

In the present paper Klein's algebraic viewpoint is extended onto arbitrary topological groups in the frames of 'Relative geometry'.

The set T_k of topological groups with k-dimensional parametric space is subdivided into topological classes. Each topological class is characterised by its only holonomic group, the geometry described in the spirit of Erlangen program. The relative geometry of two holonomic groups from T_k is determined by the difference of structure constants of commutators of their Lie algebras. Inside the topological class, group geometry is characterised by holonomy algebra which elements can serve as a quantity measure of the deviation of the deformated group geometry from the geometry of the corresponding holonomic group. As it was mentioned in the above section, the holonomy algebra is invariant under any transformation of coordinates. This together with Lie algebra invariance justifies the introduction of the concept of the group 'relative geometry'.

However, the content of group relative geometry is not completed by the investigation of holonomy algebra. Herewith a number of purely geometrical concepts and ideas, which would concretise some particular (projective, affine, etc.) geometry, must be involved.

Physical geometry, in which geometrical objects correspond to the results of physical measurement, operates with metrical concepts included through the help of a norm $\|G\|$. The intention to introduce a norm, the same for the whole topological class, demands the invariance of the norm not only under the transformation of coordinates but under the group deformations.

Free space-time is homogenous. Therefore its geometry is defined in accordance with the Erlangen program by some Lie group. A physical field breaks down space-time homogeneity as was mentioned in the Introduction. A breach of space-time homogeneity may be attributed to either a field or to a non-inertial system of reference. The latter is actually equivalent to the former because any non-inertial system of reference can be formed by a number of charges moving in a field.

The relative geometry of non-homogeneous space-time is determined by the given Lie group deformation within the corresponding topological class. Summarising the above and taking into account the traditional problems one can define the theory of relativity as a relative geometry, the physical content being introduced by the following:

Postulate: A field deforms a representation of the group of free space-time mtoions.

A field topology is here assumed to be present. It defines the topological class within which the deformation takes place.

6. *Deformation Geometry of the Poincare Group*

Consider now space-time geometry. For this purpose we shall study deformations of the Poincare group.

As is well known, the inhomogeneous group of coordinate transformations in Minkowskian space is called the Poincare group

$$x'^i = l_k{}^i x^k + y^i \tag{6.1}$$

Structurally, this group is the semidirect product $\mathscr{P} = \mathscr{L} \vee \mathscr{Y}$ of the Lorentz group $\mathscr{L}$ and the translation group $\mathscr{Y}$ with the multiplication law

$$P'.P'' = L'.L'' \vee LY'' + Y, \qquad P \in \mathscr{P}; L \in \mathscr{L}; \; Y \in \mathscr{Y} \tag{6.2}$$

Together with the natural topology, the Poincare group is the Lie group with the parametric space $\mathfrak{H}_P = \mathfrak{H}_Y \times \mathfrak{H}_L$ formed through the product of subgroup parametric spaces (Yappa, 1966). Thus the vector χ of Poincare group parametric space has ten components, the first four $\chi^i \in \mathfrak{H}_Y$ $(i = 0, 1, 2, 3)$ being the parameters of translation group and the six others $\chi^{(ik)} \in \mathfrak{H}_L$ being the parameters of Lorentz group of rotations in the plane (ik): $\chi^{(ik)} = \chi^{(ki)}$; $\chi^{(ii)} = 0$.

Also, from the definition of the semidirect group product

$$\chi^{(ik)}_{,m} \equiv 0; \qquad \chi^{m}_{,(ik)} \equiv 0 \tag{6.3}$$

i.e. the subgroups $\mathscr{L}$ and $\mathscr{Y}$ do not intersect in group $\mathscr{P}$.

As it was mentioned in Section 4, the operations of union (4.2) and intersection (4.6) govern the subgroup structure of the Poincare group. It can be easily checked that unit I of the Poincare group

$$I = E \vee O \tag{6.4}$$

is formed by the union of Lorentz group unit E and the translation group unit O. Similar correlations also take place for the generators†

$$P_\alpha = L_\alpha \vee Y_\alpha \tag{6.5}$$

Due to (6.3)

$$L_i = 0; \qquad Y_{(i\lambda)} = 0 \tag{6.6}$$

Substituting (6.5) into (2.13), in view of (6.2), we have the twist of the Poincare group‡

$$P_{[\alpha;\beta]} = \{[L_\alpha, L_\beta] + L_{[\alpha.\beta]}\} \vee \{L_{[\alpha} Y_{\beta]} + Y_{[\alpha.\beta]}\} \tag{6.7}$$

where

$$L_{[\alpha} Y_{\beta]} = L_\alpha Y_\beta - L_\beta Y_\alpha \tag{6.8}$$

Lorentz and translation subgroups, being Lie groups, are holonomic, also, they are not intersected. Thus, as expected,

$$P_{[\alpha;\beta]} = 0 \tag{6.9}$$

i.e. the Poincare group is holonomic.

Consider now a deformation $\Psi: \mathscr{P} \to \mathscr{Q}$ of the Poincare group into a topological group

$$\mathscr{Q} \ni Q = M \vee X \tag{6.10}$$

being the union of basic space (of the additive group) $\mathscr{X} \ni X$ and the group of tetrad rotations $\mathscr{M} \ni M$.

Due to the invariance of the union under deformation, the twist of the deformed Poincare group is expressed by a formula similar to (6.7):

$$Q_{[\alpha;\beta]} = \{[M_\alpha, M_\beta] + M_{[\alpha.\beta]}\} \vee \{M_{[\alpha} X_{\beta]} + X_{[\alpha.\beta]}\} \tag{6.11}$$

Nonetheless, the subgroups of the group $\mathscr{Q}$ can intersect due to deformation, so that the generators of these subgroups become intercorrelated. This phenomenon, insofar as the effects of the basic space twist $Q_{[i;k]}$ would be of interest for us in physical applications, can be written as follows:

$$M_i = \tilde{\chi}^{(mn)}_{.i} M_{(mn)} \tag{6.12}$$

Further we shall operate with the matrix representation of the Poincare group and its deformations, the Lorentz group being represented by the square (4 × 4) matrices and the translation group and basic space being represented by the column vector (4 × 1). The matrix elements will be denoted by the corresponding small letters with the indices:

$$P = \{p^a_{.b}\} = \{l^a_{.b}\} \vee \{y^a\}$$

† All group elements and generators are considered locally at some point of parametric space.

‡ This means that from (2.6) and (6.2) it follows: $dP^{\cdot}.dP^{\cdot} = dL^{\cdot}.dL^{\cdot} \vee dL^{\cdot}.dY^{\cdot}$.

In particular (6.12) 'in coordinates' will be

$$(m^a_{.b})_i = \bar{\chi}^{(mn)}_{,i}(m^a_{.b})_{(mn)} \tag{6.13}$$

since X_k form the full basis in vector space $\mathscr{X}$, the one-valued expansion takes place

$$\bar{\chi}^{(mn)}_{,i} = (x^a)_i\,\theta^{(mn)}_a \tag{6.14}$$

which after substitution into (6.13) gives

$$(m^a_{.b})_i = [\theta^{(mn)}_c(m^a_{.b})_{(mn)}](x^c)_i \tag{6.15}$$

In the matrix form the last correlation is:

$$M_i = X_i^{\tau}\Gamma \tag{6.16}$$

where $\Gamma(x)$ is a three-dimension (cubic) matrix connectivity in which coordinates are given by square brackets in (6.15); X_i^{τ} is the transpose matrix X_i (the row 1×4).

The substitution of (6.12) into (6.16) gives two expressions for the twist of the basic space of the deformed Poincare group:

$$Q_{[i;k]} = \bar{\chi}^{(mn)}_{,i}\,\bar{\chi}^{(rs)}_{,k}\,M_{[(mn);(rs)]} \vee M_{(mn)}\,\bar{\chi}^{(mn)}_{,[i}\,X_{k]} + X_{[i,k]} \tag{6.17}$$

$$Q_{[i;k]} = \{X^{\tau}_{[i}\Gamma X^{\tau}_{k]}\Gamma + X^{\tau}_{[i,k]}\Gamma + X^{\tau}_{[i}\Gamma_{,k]}\} \vee \{X^{\tau}_{[i}\Gamma X_{k]} + X_{[i,k]}\} \tag{6.18}$$

where $\Gamma_{,k} = (\partial/\partial\bar{\chi}^k)\Gamma$.

Now let us introduce an invariant norm (an interval) into the topological class of the Poincare group as follows

$$ds = \|dQ\| = \sqrt{(dX^{\tau}G\,dX)} \tag{6.19}$$

where $G = \{g_{ik}(\bar{\chi})\}$ is a metrical matrix (4×4) satisfying the equation

$$G = M^{\tau}HM \tag{6.20}$$

being the matrix of pseudoeuclidean square form

$$H = \mathrm{diag}(1, -1, -1, -1) \tag{6.21}$$

which serves also as the metric matrix of the Poincare group in the canonical coordinates.

An infinitesimal transformation in the rotation group: $M' = dM.M$ is compensated for by an infinitesimal transformation of the metrical matrix: $G' = G + dG$, so that $M'^{\tau}G'M' = H$ or

$$(M^{\tau} + M^{\tau}dM^{\tau})H(dM.M + M) = G + dG \tag{6.22}$$

Using (6.20), we have with first-order accuracy

$$M^{\tau}(dM^{\tau}H + H\,dM)M = dG \tag{6.23}$$

Taking into account (6.20) and (A.16), this reduces to

$$dG = dM.G + G.dM \tag{6.24}$$

whereby, denoting $G_{,\alpha} = (\partial/\partial\chi^\alpha)G$, we obtain

$$G_{,\alpha} = M_\alpha{}^\tau G + GM_\alpha \tag{6.25}$$

which, after substituting (6.16), gives

$$G_{,i} = \Gamma^\tau X_i G + GX_i{}^\tau \Gamma \tag{6.26}$$

Now one can introduce the concept of a derivative in the topological class of the Poincare group

$$\dot{Q} = \frac{dQ}{ds} \tag{6.27}$$

Owing to the linearity, one has

$$\dot{Q} = \dot{M} \vee \dot{X}; \qquad \ddot{Q} = \ddot{M} \vee \ddot{X} \tag{6.28}$$

where the corresponding derivatives with respect to the group norm are denoted by dots: $\dot{X} = (dX/ds)$. In terms of derivatives formula (6.24) will be

$$\dot{G} = \dot{M}^\tau G + G\dot{M} \tag{6.29}$$

and (6.17) will be written

$$\dot{M} = \dot{X}^\tau \Gamma \tag{6.30}$$

Also from (A.15) and (A.16) we have

$$\dot{M}M = \dot{M}M; \qquad \dot{M}^{-1} = -M^{-1}\dot{M}M \tag{6.31}$$

As it was mentioned in the last part of Section 3, homeomorphism $\Psi: \mathscr{P} \to \mathscr{Q}$ can be expanded into the horizontal and vertical components:

$$Q(\bar{\chi}) = S(\bar{\chi}).P(\chi) \tag{6.32}$$

Following the usual notation we shall from now on use the basic representation in which

$$X_i = \{\delta_i{}^k\}; \qquad dx^i = d\chi^i \tag{6.33}$$

To provide this in view of (3.23), (4.2), (6.2) and (6.32), it is necessary and sufficient to put the deformation operator as

$$S = M \vee O \tag{6.34}$$

where

$$M = \{m^k_{.i}\}; \qquad \varphi_i{}^k m_k{}^m = \delta_i{}^m \tag{6.35}$$

In the basic representation, (6.26) gives

$$\frac{\partial g_{km}}{\partial x^i} = g_{kn}\gamma^n_{i.m} + \gamma^n_{i.k} g_{nm} \tag{6.36}$$

This is the well-known correlation of Riemannian geometry between the metric tensor g_{ik} and the Christoffel symbols $\gamma^m_{i.k}$.

The twist of the deformated Poincare group can be written as

$$Q_{[i;k]} = R_{ik} \vee T_{ik} \tag{6.37}$$

whereas having substituted (6.33) into (6.18) we have in coordinates

$$\begin{aligned} (r^{a}_{.b})_{ik} &= \gamma^{a}_{i.m}\gamma^{m}_{k.b} - \gamma^{a}_{k.m}\gamma^{m}_{i.b} + \frac{\partial \gamma^{a}_{i.b}}{\partial x^{k}} - \frac{\partial \gamma^{a}_{k.b}}{\partial x^{i}} \\ (t^{a})_{ik} &= \gamma^{a}_{i.k} - \gamma^{a}_{k.i} \end{aligned} \tag{6.38}$$

This, taking into account geometrical sense and functional correlations (Schouten, 1951), makes it possible to identify $(r^{a}_{.b})_{ik}$ and $(t^{a})_{ik}$ with a curvature tensor r^{a}_{ikb} and a torsion tensor t^{a}_{ik}, correspondingly. As a result of this, the generators $Q_{[i;k]}$ of holonomy algebra in (6.37) may be considered as the union of curvature generators R_{ik} and torsion generators T_{ik}. The substitution of (6.33) into (6.17) gives the following expressions for the curvature tensor and the torsion tensor:

$$\begin{aligned} (r^{a}_{.b})_{ik} &= \bar{X}^{(mn)}_{.[i}\bar{X}^{(rs)}_{.k]}(nr^{a}_{.b})_{[(mn):(rs)]} \\ (t^{a})_{ik} &= \bar{X}^{(mn)}_{.i}(m^{a}_{.k})_{(mn)} - \bar{X}^{(mn)}_{.k}(m^{a}_{.i})_{(mn)} \end{aligned} \tag{6.39}$$

The physical application of the theory will be considered in the next section. Nevertheless, the results obtained above already allow one to conclude that the relative geometry of the Poincare group deformations can serve as the mathematical base for Einstein's theory of relativity.

7. *Accelerated Reference System*

An element P of the Poincare group can be interpreted as a transformation (6.1) of initial basis, y^{i}, being the coordinates of the initial basis origin in reference to the new one and $l^{i}_{.k}$ being the projection of ith axis of the new basis on to the kth axis of the initial one.

A trajectory $\chi^{\alpha} = \chi^{\alpha}(s)$ determinates motion of the basis $L[\chi(s)] \vee Y[\chi(s)]$ in the Poincare group. Define a rotating reference system with the help of basis rotating without deformation in the plane (12) by the law

$$\chi^{0} = s; \qquad \chi^{(ik)} = \delta^{(ik)}_{(12)}\frac{\omega}{c}s; \qquad \chi^{i} = \text{const}\,(i \neq 0) \tag{7.1}$$

where ω is the angular velocity. This corresponds to a one-parameter subgroup of the Poincare group:

$$P(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\omega}{c}s & -\sin\frac{\omega}{c}s & 0 \\ 0 & \sin\frac{\omega}{c}s & \cos\frac{\omega}{c}s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} s \\ \chi^{1}\cos\frac{\omega}{c}s - \chi^{2}\sin\frac{\omega}{c}s \\ \chi^{1}\sin\frac{\omega}{c}s + \chi^{2}\cos\frac{\omega}{c}s \\ \chi^{3} \end{pmatrix} \tag{7.2}$$

The rotating reference system should naturally be called a deformed Poincare group $\mathcal{Q} = \Psi(P)$ in that in it the basis (7.2) does not rotate, i.e.

$$\Psi[P(s)] = Q(s) = E \vee X(s) \tag{7.3}$$

Comparison of (7.2) and (7.3) with (6.32) gives the following deformation operator:

$$S(s) = \Lambda(s) \vee 0 \tag{7.4}$$

$$\Lambda(s) = L^{-1}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\omega}{c}s & \sin\frac{\omega}{c}s & 0 \\ 0 & -\sin\frac{\omega}{c}s & \cos\frac{\omega}{c}s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.5}$$

$$M(s) = E; \qquad X(s) = L^{-1}(s).\, Y(s) = \begin{pmatrix} s \\ \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix} \tag{7.6}$$

Let us choose now in the rotating reference system such a coordinate system where: (i) the motion law (7.1) of the fundamental basis is kept; (ii) the basic representation (6.33) is provided. Having (3.12), (6.32) and (7.5) it is easily seen that both conditions satisfied at such a coordinate mapping as:

$$\begin{aligned} d\chi^1 &= d\bar{\chi}^1 \cos\frac{\omega}{c}\bar{\chi}^0 - d\bar{\chi}^2 \sin\frac{\omega}{c}\bar{\chi}^0 \\ d\chi^2 &= d\bar{\chi}^1 \sin\frac{\omega}{c}\bar{\chi}^0 + d\bar{\chi}^2 \cos\frac{\omega}{c}\chi^0 \end{aligned} \tag{7.7}$$

where $d\chi^\alpha = d\bar{\chi}^\alpha$ $(\alpha \neq 1,2)$.

Deformation (7.3) and (7.8) of the Poincare group into the rotating reference system causes an intersection of the rotation group with the basic space. In fact, though in the case considered we do have:

$$X_i(\bar{\chi}) = Y_i(\chi); \qquad M_{(ik)}(\bar{\chi}) = L_{(ik)}(\chi) \tag{7.9}$$

but due to (3.23) an additional generator appears in the rotating reference system:

$$M_0(\bar{\chi}) = \Lambda_0(\chi^0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega}{c} & 0 \\ 0 & \frac{\omega}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.10}$$

which can be written as

$$M_0(\bar{\chi}) = \bar{\chi}^{(12)}_{.0} L^0_{(12)}; \qquad \bar{\chi}^{(12)}_{.0} = -\frac{\omega}{c} \tag{7.11}$$

where $\bar{\chi}^{(12)}_{.0}$ is the intersection of the rotation group in plane (12) with the time translation in the rotating reference system; $L^0_{(12)}$ is the infinitesimal operator of the Lorentz group of rotations in plane (12). The origin of the minus in (7.11) is connected with the compensating nature of the generator $M_0(\bar{\chi})$.

Substituting (7.11) into (6.17) one can see that curvature R_{ik} in the rotating reference system is equal to zero, but torsion T_{ik} has components different from zero where the twist of the rotating reference system is in the form

$$Q_{[i;k]}(\bar{\chi}) = 0 \vee -\frac{\omega}{c} L^0_{(12)} \delta_{[i} X_{k]} = \text{const} \tag{7.12}$$

governing the curvature tensor

$$t^2_{10} = -t^2_{01} = t^1_{02} = -t^1_{20} = \frac{\omega}{c} \tag{7.13}$$

The transition to the rotating reference system does not change the metrics because of (6.25). Besides, thanks to the above-mentioned fact, space-time is left flat in the rotating reference system.

The above example, considered in detail, allows one to formulate the general definition of the reference system. So in future a homeomorphism $\mathcal{Q} = \mathcal{M} \vee \mathcal{X}$ of the Poincare group will be identified with the reference system formed by the fundamental basis $E \vee \mathcal{X}$ with the help of the deformation operator $S = \Lambda \vee 0$. The formation of the group being given as $Q = Q(x)$ for example, through a fundamental basis motion law: $\chi^a = \chi^a(s)$ invariant under the homeomorphism, one may choose a coordinate system. If the deformer Λ belongs to the Lorentz group, then the corresponding deformation does not curve the basic space holding pseudo-euclididenean metrics in it. It is convenient to use the canonical coordinate system, in this case putting

$$d\bar{\chi}^i = dx^i; \qquad d\bar{\chi}^{(ik)} = d\chi^{(ik)} \tag{7.12}$$

The totality of bases moving in the Poincare group by the law

$$\chi^{(ik)} = \text{const} \tag{7.13}$$

forms an inertial reference system, the Poincare group itself being the inertial reference system with the fundamental basis: $\chi^{(ik)} = 0$. It can be readily shown that the homeomorphisms to the inertial reference system are holonomic, so that all of i.r.s. are isomorphic on the basis of Theorem 3.

One can now construct a uniformly accelerated reference system to give one more example. As it is well known, the motion of a particle in the inertial reference system according to the law

$$Y = \left(\frac{c^2}{a}\,\mathrm{sh}\frac{a}{c^2}s,\ \frac{c^2}{a}\left(\mathrm{ch}\frac{a}{c^2}s - 1\right), 0, 0\right) \tag{7.14}$$

is called the uniformly accelerated (Landau, 1961) or hyperbolic (Rogozhin, 1966) motion, where a is the Newtonian acceleration of particle; s is an interval along its world line, the particle resting at first (when $s = 0$) at the origin of the initial basis of the Poincare group.

The equation (7.14) may be considered as a projection of the one-parameter subgroup of the Poincare group:

$$P(s) = \begin{pmatrix} \mathrm{ch}\frac{a}{c^2}s & \mathrm{sh}\frac{a}{c^2}s & 0 & 0 \\ \mathrm{sh}\frac{a}{c^2}s & \mathrm{ch}\frac{c^2}{a}s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} \frac{c^2}{a}\,\mathrm{sh}\frac{a}{c^2} \\ \frac{c^2}{a}\left(\mathrm{ch}\frac{a}{c^2} - 1\right) \\ 0 \\ 0 \end{pmatrix} \tag{7.15}$$

onto the translation group. In fact,

$$P(s+t) = L(s+t) \vee Y(s+t) = P(s).P(t) \tag{7.16}$$

follows from (7.15) confirming the above statement. The corresponding trajectory of motion of the basis $P(s)$ in parametric space is found from the equations:

$$dP(s) = P(s+ds).P^{-1}(s) - I = P_\alpha d\chi^\alpha \tag{7.18}$$

whence having $P^{-1} = L^{-1} \vee -L^{-1}Y$, knowing the generators $P_\alpha = L_\alpha \vee Y_\alpha$ and accounting independence of $d\chi^\alpha$ one obtains

$$\chi^{(ik)}(s) = \delta^{(ik)}_{(01)}\frac{a}{c^2}s; \qquad \chi^i(s) = \delta_0{}^i s \tag{7.19}$$

which corresponds to 'a rotation' of the basis in the plane (01).

The uniformly accelerated reference system is to be defined now through the Poincare group deformation which acts by formula (7.3). Further considerations should be analogous to those for the rotating reference system. The only difference is that instead of (7.5) we have the following deformer:

$$A(s) = L^{-1}(s) = \begin{pmatrix} \mathrm{ch}\frac{a}{c^2}s & -\mathrm{sh}\frac{a}{c^2}s & 0 & 0 \\ -\mathrm{sh}\frac{a}{c^2}s & \mathrm{ch}\frac{a}{c^2}s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.20}$$

One should put $s = \chi^0$ into (7.20) for the fundamental basis to move in the constructed reference system by the same law (7.19). As a result, the following mapping of coordinates between the (parametric spaces of the) inertial reference system and the uniformly accelerated reference system is obtained:

$$d\chi^0 = d\bar{\chi}^0 \operatorname{ch}\frac{a}{c^2}\bar{\chi}^0 + d\bar{\chi}^1 \operatorname{sh}\frac{a}{c^2}\bar{\chi}^0$$

$$d\chi^1 = d\chi^0 \operatorname{sh}\frac{a}{c^2}\bar{\chi}^0 + d\bar{\chi}^1 \operatorname{ch}\frac{a}{c^2}\bar{\chi}^0 \tag{7.21}$$

where $d\chi^\alpha = d\bar{\chi}^\alpha$ $(\alpha \neq 0, 1)$. As for the generators of the uniformly accelerated reference system they are easily checked to be

$$M_0(\bar{\chi}) = \bar{\chi}^{(01)}_{,0} L^{(01)}_{,0}; \qquad \bar{\chi}^{(01)}_{,0} = -\frac{a}{c^2} \tag{7.22}$$

as an addition to (7.9). Since one can calculate the twist of the uniformly accelerated reference system:

$$Q_{[i;k]}(\bar{\chi}) = 0 \vee -\frac{a}{c^2} L^0_{(01)} \delta^0_{[i} X_{k]} = \text{const} \tag{7.23}$$

The infinitesimal operator value of the Lorentz group of 'rotation' in the plane (01) when substituted into (7.23) gives the non-zero components of the torsion tensor as:

$$t^0_{01} = -t^0_{01} = \frac{a}{c^2} \tag{7.24}$$

This result had been obtained by the author earlier, on quite the other grounds (Rogozhin, 1965). In the same work there was considered the physical sense of space-time torsion in the uniformly accelerated reference system.

The conclusion about the appearance of torsion in the rotating reference system may have been obtained here for the first time, though it seems an almost obvious consequence from the analogy between the mentioned reference systems, both forming by the rotating basis.

We have also considered various combinations of basis torsions forming screw, helical and some other reference systems. The results will be published in another paper; the purpose of the given paragraph was to demonstrate in simple examples the efficiency of the apparatus of the group deformation theory developed above. As a result of the torsion, the mapping of coordinates between the inertial reference system and the various accelerated reference systems should not be integrable, providing invariance of dynamical effects under arbitrary transformations of coordinates within a reference system. This fact attaches physical sense to our definitions of the accelerated reference systems, unlike the canonical definitions (Landau, 1961; Rogozhin, 1966).

8. *Classical Fields*

Physical fields acting within space-time are called classical fields. In this case the Postulate of Section 4 is formulated as follows: *A classical field deforms the motion group of free space-time.*

Insofar as the Poincare group is just the group of motion of free space-time, the classical fields deform the Poincare group so as to act on to space-time.

In accordance with the previous paragraph, geometry of a classical field is determined by some reference system resulting from the inertial reference system with the help of deformation (6.23) and (6.35) as

$$Q(\bar{\chi}) = S(\bar{\chi}).P(\chi) = [\Lambda(\bar{\chi}) \vee 0].[E \vee Y(\chi)] \tag{8.1}$$

where the deformer $\Lambda(\bar{\chi})$ must have been given by the field equations or by the equations of charge motion in the field.

Let a one-parameter subgroup of the Poincare group homeomorphism $\mathcal{Q} = \mathcal{M} \vee \mathcal{X}$ be named the world line of a particle if: (i) the trajectory $\bar{\chi}(s)$ is determined on the translation parametric subspace, i.e. $\bar{\chi}(s) \equiv \bar{\alpha}(s) \in \mathfrak{H}_x$; (ii) the corresponding norm $\|Q[\bar{\chi}(s)]\|$ is a real number.

The world line of a particle is invariant under deformation (8.1) as follows from the properties (6.35) of the deformer. But if a particle moves in the inertial reference system by the law:

$$P[\alpha(s)] = E \vee Y[\alpha(s)] \tag{8.2}$$

then motion of the particles in an arbitrary reference system would be described by the equations

$$Q[\bar{\alpha}(s)] = M[\bar{\alpha}(s)] \vee X[\bar{\alpha}(s)] \tag{8.3}$$

in which orientation of the local basis relative to the initial is accounted and where through (8.1)

$$M[\bar{\alpha}(s)] = \Lambda[\bar{\alpha}(s)]; \qquad X[\bar{\alpha}(s)] = \Lambda[\bar{\alpha}(s)].\, Y[\alpha(s)] \tag{8.4}$$

We shall use the basic representation (6.32) later, which allows the possibility of using the interval (6.19) along the world line of a particle as the parameter s. The marks of values of the parameters will usually be omitted when all quantities have been taken at the same values of the interval.

Differentiating (8.1) with respect to the interval in accordance with (3.18) one obtains

$$\dot{X} = \Lambda \dot{Y} \tag{8.5}$$

This equation connects the velocity $\dot{Y}$ of a particle relative to the inertial reference system and velocity $\dot{X}$ relative to an arbitrary reference system. Repeated differentiation according to (3.19) and with respect to (6.31) gives the equation for acceleration of a particle in the above reference systems:

$$\ddot{X} = \Lambda \ddot{Y} + \dot{\Lambda} \dot{Y} \tag{8.6}$$

Let us call now the particles uniformly moving in their local basis, basic particles of the given reference system (for the given deformation of the Poincare group). In other words,

$$\dot{X} = \text{const}; \qquad \ddot{X} = 0 \tag{8.7}$$

along the world line of the basic particles. Equation (8.6) for the basic particles of the $\mathscr{Q}$ reference system will be

$$\ddot{Y} + \dot{\Lambda} \dot{Y} = 0 \tag{8.8}$$

In particular, since for the Poincare group in canonical coordinates $L = 0$ along the world lines of any particles the 'free' particles, for which $\ddot{Y} = 0$, serve as the basic particles of the inertial reference system. In general the rotation (basis rotations) subgroup of the Poincare group arbitrary deformation 'compensates' the acceleration of its own basic particles in the inertial reference system according to (8.8).

Interaction of a (classical) particle with given (classical) field are determined by the (specific) charge ϵ. Let us use a system of units where:

(1) $\epsilon = 0$ for the particles which do not interact with the field;

(2) the particles with unit-specific charge are basic particles for the corresponding field (more precisely, for a Poincare group deformation caused by the field).

If $\mathscr{Q}$ is the Poincare group deformation caused by the field then equation (8.6) hold for an arbitrary particle in the presence of the field. To introduce into the equation the concept of a charge satisfying both demands, it is enough to postulate the following equation of motion relative to the inertial reference system

$$\ddot{Y} + \epsilon \dot{\Lambda} \dot{Y} = 0 \tag{8.9}$$

First of all, when $\epsilon = 0$, (8.9) gives $\ddot{Y} = 0$ describing the motion of a free particle.

Secondly, putting in (8.9) $\epsilon = 1$, we obtain equation (8.8) for a basic particle of the reference system $\mathscr{Q}$. The substitution of (8.9) into (8.6) with respect to (8.5) and (6.31) gives the motion equation of the charge relative to the reference system proper for the field:

$$\ddot{X} = (1 - \epsilon) \dot{\Lambda} \dot{X} \tag{8.10}$$

where, as well as in (8.9), all quantities are determined at some interval value along the world line of the charge.

Now, it is immediate that the electromagnetic field would be described by our formalism, if we take $\epsilon = e/mc^2$ and write the following deformer derivative:

$$\acute{\Lambda} = -G^{-1}.F \tag{8.11}$$

in (8.1), where $F = \{f_{ik}\}$ is the matrix formed by the electromagnetic field tensors f_{ik}, where antisymmetry gives

$$\acute{\Lambda}^{\tau} = FG^{-1} = -G\acute{\Lambda}G^{-1} \tag{8.12}$$

From equation (6.29) it follows that along the world line of the charge the field metric remains unchangeable.

Due to this, one can introduce in the whole field the Galilean coordinates: $G = H$ in which the unit charge 'rests', i.e. we have (8.7) for $\epsilon = 1$ and in the basic representation

$$\dot{\bar{\chi}}^i(s) = \dot{\bar{\alpha}}^i(s) = \dot{x}^i(s) = \delta_0{}^i; \qquad \acute{\Lambda}(s) = \Lambda_0(\bar{\alpha}) \tag{8.13}$$

Substituting (8.12) and (8.13) into (6.11) we obtain the twist

$$Q_{[i;k]} = 0 \vee T_{ik} \tag{8.14}$$

with zero curvature, $R_{ik} = 0$, and torsion generators different from zero:

$$T_{ik} = -\acute{\Lambda}\delta^0_{[i}X_{k]} \tag{8.15}$$

This corresponds to the torsion tensors:

$$t^i_{k0}(\bar{\alpha}) = -t^i_{0k}(\bar{\alpha}) = f_k{}^i(\alpha) \tag{8.16}$$

The mapping of coordinates: $\chi^i \to \bar{\chi}^k$ between the inertial reference system and proper reference system of the electromagnetic field may be found from equations (8.6), (8.11) and (8.13). This mapping is non-integrable due to torsion of the basic space. The latter is in the spirit of general relativity which proclaimed physical equivalency of reference systems connected between each other by transformations (holonomic mapping) of coordinates.

Generator $\Lambda_0(\bar{\alpha})$ of the electromagnetic field deformer because of (8.11) and (8.13) can be written as

$$\Lambda_0(\bar{\alpha}) = \bar{\chi}^{(ik)}_{.0} L^0_{(ik)} \tag{8.17}$$

where

$$\bar{\chi}^{(0k)}_{.0} = -E_k; \qquad \bar{\chi}^{(12)}_{.0} = H_3; \qquad \bar{\chi}^{(13)}_{.0} = H_2; \qquad \bar{\chi}^{(23)}_{.0} = H_1 \tag{8.18}$$

Thus intensities of the electromagnetic field directly determine an intersection of the rotation group with the basic space of the proper group for the field reference system, the fundamental basis of which rotate in the Poincare group without deformation (without their curving). As a result, the electromagnetic field manifests itself in the a torsion of space-time. The connection between electromagnetism and torsion was mentioned in many works (Well, 1920; Emstein, 1930; Eddington, 1922; Schrödinger, 1950; Arbusov & Fillipov, 1967; Rodicev, 1968), the author's work (Rogozhin, 1968) included. Nevertheless, the present result (8.16) differs from those given earlier, though it seems almost obvious.

To obtain the motion equation of a particle in the other classical, gravitational, field we shall proceed from the following experimental facts:

(1) All the bodies, regardless of their nature, have equal specific charges i.e. all the bodies fall with the same acceleration)—experiments of Galilei and Eötvos.

(2) The proper reference system of a particle moving in the gravitational field is locally inertial (Einstein's interpretation of experiments in falling lifts).

Relying on these facts, for all the particles in the gravitational field put

$$\bar{Y} = 0 \tag{8.19}$$

which after substitution into (8.6) accounting (8.5) and (6.31) gives

$$\ddot{X} = \Lambda \dot{X} \tag{8.20}$$

Whence, due to (6.29), we have

$$\ddot{X} = \dot{X}^{\tau} \Gamma \dot{X} \tag{8.21}$$

being the well-known Einstein's equations of motion when written in coordinates.

If following to Einstein the connection matrix is assumed symmetric

$$\Gamma^{\tau} = \Gamma \tag{8.22}$$

then the formulas (6.38) state that gravitation manifests itself as curvature of space-time without its torsion. It should be noted that equation (8.19) can be obtained from (8.10) immediately if in that equation one puts

$$\epsilon = 0 \tag{8.23}$$

Assumption (8.23) about zero 'gravitational charge' for all the bodies does not in the least contradict the facts mentioned. Moreover it is likely to follow from them.

Hence, despite their different nature, the influence of the electromagnetism and gravitation on space-time can be looked at as the general property of these fields which show themselves up in one or other distortions of space-time symmetries. In this sense one can speak about a unified classical, electromagnetic-gravitational, field under influence of which space-time undergoes curvature and torsion. In geometrical interpretation the unified field is a field of local basis above space-time. The reciprocal twist of basis belonging to adjacent space-time points can be divided into two parts, deformation and rotation, which would correspond to the division of the unified classical field into two components: gravitation and electromagnetism.

9. *Summary*

1. The concept of 'holonomy' can be effectively used in the theory of arbitrary topological groups.
2. Group deformation is described by nonlinear mapping of holonomy algebra (i.e. by non-holonomic mapping of group formation).
3. The operations of union and intersection of subgroups define the subgroups structure which can be deformed as well.
4. Relative geometry of topological groups describes a deviation from geometry of the holonomic group of the same topological class.
5. Relative geometry of a deformed Poincare group is characterised by curvature and torsion caused by breakdown of holonomicity in the Lorentz subgroup and translations corresponding.
6. The basic space possesses torsion in the rotated and accelerated reference system.
7. The classical fields deform the Poincare group.
8. Holonomy algebra, found from the equations of motion, allows consideration of gravitation as a curvature and electromagnetism as a torsion of space-time.
9. The different nature of these fields is evident immediately from the equations of motion, since the electromagnetic field deals with relative motion of various charges, but the gravitational field concerns relative motion of free bodies.
10. Nevertheless, the unification of classical fields into the unified theory, with the help of our formalism, becomes as natural as merging of the rotation group and the translation group into a unified group of inhomogeneous transformations.
11. The results obtained offer possibilities to describe an arbitrary (not only classical) field in the group-theoretical language through the terms of relative geometry (holonomy algebra, metrics, etc.) of corresponding deformation of topologically given Poincare group representation.

Appendix

Discussion of group differentiation rules used in the text:
From the definition of the group differential

$$dG(\chi) = G(\chi + d\chi)\, G^{-1}(\chi) \tag{A.1}$$

it follows that

$$G(\chi + d\chi) = dG(\chi) \cdot G(\chi) \tag{A.2}$$

The second group differential is defined analogously.

$$\delta dG(\chi) = dG(\chi + \delta\chi)\, dG^{-1}(\chi) \tag{A.3}$$

It is useful to apply the truncated differentials of the group:

$$dG(\chi) = E + dG(\chi) \tag{A.4}$$

$$\delta dG(\chi) = E + \delta dG(\chi) \tag{A.5}$$

which for small $d\chi$ may be put as:

$$dG(\chi) \approx G_i(\chi)\, d\chi^i \tag{A.6}$$

whence

$$\delta dG \approx \delta(G_i\, d\chi^i) = G_{i,k}\, d\chi^i\, \delta\chi^k + G_i\, \delta d\chi^i \tag{A.7}$$

where $G_{i,k}$ should be called a derivative of the generators in the direction k.
With accuracy up to the second-order we have

$$\delta G . dG = E + \delta G + dG + \delta G . dG \tag{A.8}$$

$$\delta dG . d\delta G = E + \delta dG + d\delta G \tag{A.9}$$

whence to the same accuracy one obtains

$$dG^{-1} = E - dG + dG^2 \tag{A.10}$$

$$\delta dG^{-1} = E - \delta dG \tag{A.11}$$

Let us now use $K(\chi)$, $F(\chi)$, $G(\chi)$ as elements of a group defined on a parametric space so that

$$K(\chi) = F(\chi) . G(\chi) \tag{A.12}$$

By the definition (A.1):

$$dK(\chi) = [F(\chi + d\chi) . G(\chi + d\chi)] . [F(\chi) . G(\chi)]^{-1} \tag{A.13}$$

whence accounting (A.2) and (A.4) we obtain a formula for differential of product

$$dK = dF + F dG F^{-1} \tag{A.14}$$

If $F = G^{-1}$ then $K \equiv E$ and

$$dG^{-1} = -G^{-1}\, dG G \tag{A.15}$$

yielding (6.31) in view of (A.10) with first-order accuracy.
Repeated differentiation of (A.3) results in a formula:

$$\delta dK = \delta dF + F\{\delta dG + [\delta F, dG]\} F^{-1} \tag{A.16}$$

and $[\delta F, dG] = \delta F . dG - dG . \delta F$.

References

Arbusov, B. A. and Fillipov, A. T. (1967). *Soviet Physics: JETP*, **52**, 1092.
Bishop, R. L. and Critenden, R. I. (1964). *Geometry of Manifolds*. Academic Press.
Cartan, E. (1927a). *Journal de mathématiques pures et appliquées*, **6**, 1.
Cartan, E. (1927b). *L'Enseignement Math.*, 200.

Chevalley, C. (1946). *Theory of Lie Groups.*
Chew, G. F. (1961). *The S-matrics Theory of Strong Interactions.* N.Y.
Eddington, A. S. (1922). *Mathematical Theory of Relativity.*
Einstein, A. (1930). *Annales de l'Institut Henri Poincaré*, **1**, 1.
Gerstenhaber, M. (1964). *Annals of Mathematics*, Vol. 79, No. 1.
Ihonu, E. and Wigher, E. P. (1953). *Proceedings of the Natural Science Association of Staten Island*, **39**, 510.
Jacobson, N. (1961). *Lie Algebras.* Wiley.
Klein, F. (1893). *Mathematische Annalen*, **43**, 261.
Landau, L. D. and Lifshitz, E. M. (1961). *Classical Theory of Fields.*
Levy-Nahas, M. (1967). *Journal of Mathematical Physics*, **9**, 1211.
Lichnerowicz, A. (1955). *Theorie Globale des Connexions et des Groups d'Holonomie.* Roma.
Neimark, M. A. (1963). *Linear Representation of the Lorentz Group.*
Nomizu, K. (1956). *Lie Groups and Differential Geometry.* Tokyo.
Pontryagin, L. S. (1959). *Topological Groups.*
Rodicev, V. P. (1968). *Abstracts 5th International Conference on Gravitation and Relativity, Tbilisi.*
Rogozhin, Yu. A. (1965). *Abstracts of II Soviet Gravitation Conference. Tbilisi.*
Rogozhin, Yu. A. (1966). *Preprint Institute of Physics and Energetics, No. 59, Obninsk, USSR.*
Rogozhin, Yu. A. (1968). *Abstracts 5th International Conference on Gravitation and Relativity, Tbilisi.*
Sakyrai, J. J. (1960). *Annals of Physics*, **11**, 1.
Sadowsky, L. E. (1968). *Uspekhi Fizicheskikh nauk.* Vol. 34, No. 3.
Schrödinger, E. (1950). *Space–Time–Structure.* Cambridge.
Schouten, J. A. (1951). *Tensor Analysis for Physicists.* Oxford.
Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics and all that.* Benjamin.
Suzuki, M. (1956). *Structure of a Group and the Structure of its Lattice of Subgroups.*
Utiyama, R. (1956). *Physical Review*, **101**, 1597.
Well, H. (1920). *Raum-Zeit-Materie.* Berlin.
Yappa, Yu. Ya. (1966). 'Physics and Chemistry', *Vestnik Leningrad University*, No. 22.